ON ENESTRÖM –KAKEYA THEOREM

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Abstract: In this paper we obtain some interesting Eneström-Kakeya type theorems concerning the location of zeros of polynomials. Our results extend and generalize Some well known results by putting less restrictive conditions on coefficients of polynomials.

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1.Introduction and statement of results:

The following elegant result which is well known in the theory of the distribution of the zeros of a polynomial is due to Eneström and Kakeya[6].

Theorem A: If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$, is a polynomial of degree n,such that

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0, \qquad (1)$$

then all the zeros of P(z) lie in |z|≤1. This is a beautiful result but it is equally limited in scope as the hypothesis is very restrictive. In the literature [1,3,5,7,8], there exists some extensions and generalizations of Eneström-Kakeya Theorem .

Recently Aziz and Zargar[2], relaxed the hypothesis of Theorem A in several ways and proved the following results.

Theorem B: If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ is a polynomial of degree n such that for some k≥1.

$$ka_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0 \qquad (2)$$

then P(z) has all its zeros in $|z+k-1| \leq k$

Theorem C: If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ is a polynomial of degree n≥2, such that either

$a_n \geq a_{n-2} \geq .... \geq a_3 \geq a_1 > 0,$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0,$ if n is odd

or

$a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0,$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq a_1 > 0,$ if n is even,

then all the zeros of P(z) lie in the circle

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1}}{a_n} \qquad (3)$$

Theorem B is an interesting extension of Theorem A.

In this paper we shall first present the following extension of Theorem C analogous to Theorem B which among other things include Theorem A as a special case.

Theorem 1.1: If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ is a polynomial of degree n≥2 `such that for some k≥1, either

$ka_n \geq a_{n-2} \geq .... \geq a_3 \geq a_1 > 0$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$,if n is odd

or

$ka_{n-1} \geq a_{n-3} \geq .... \geq a_2 \geq a_0 > 0$ and $a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq a_1 > 0$, if n is even then all the zeros of P(z) lie in the region

$$|z+\alpha||z+\beta| \le (k+\frac{a_{n-1}}{a_n})$$

where α, β are the roots of the quadratic

$$z^2+\frac{a_{n-1}}{a_n}z+k-1=0 \qquad (4)$$

Taking $a_{n-1}=2a_n\sqrt{k-1}$ and noting that the quadratic $z^2+2\sqrt{(k-1)}z+k-1=0$ has two equal roots each is equal to $-\sqrt{(k-1)}$, we get the following:

Corollary 1: If $P(z)=a_nz^n+a_{n-1}z^{n-1}+....+a_1z+a_0$ is a polynomial of degree n≥2 such that for some k≥1, either

$ka_n \ge a_{n-2} \ge .... \ge a_3 \ge a_1 > 0$ and $2a_n\sqrt{k-1}=a_{n-3} \ge .... \ge a_2 \ge a_0 > 0,$ if n is odd or (5)

$ka_{n-1} \ge a_{n-3} \ge .... \ge a_2 \ge a_0 > 0$, and $2a_n\sqrt{k-1}=a_{n-1} \ge a_{n-3} \ge .... \ge a_3 \ge a_1 > 0,$

if n is even.

then all the zeros of P(z) lie in the circle

$$\left|z+\sqrt{k-1}\right| \le \left(k+2\sqrt{k-1}\right)^{\frac{1}{2}} \qquad (6)$$

Applying Corollary 1 to the polynomial

$$F(z)=b_{2n}z^{2n}+b_{2n-1}z^{2n-1}+....+b_1z+b_0,$$

of even degree 2n, we get

Corollary 2: if

$$F(z)=\sum_{j=0}^{2n} b_jz^j$$

is a polynomial of even degree 2n such that $kb_{2n} \ge b_{2n-2} \ge .... \ge b_2 \ge b_0 > 0,$ and $(k-1)b_{2n}=b_{2n-1} \ge b_{3n-3} \ge .... \ge b_3 \ge b_1 > 0$, then all the zeros of P(z) lie in

$$\left|z+\sqrt{k-1}\right| \le \left(k+2\sqrt{k-1}\right)^{\frac{1}{2}}$$

Remark 1: Corollary 2 includes Eneström-Kakeya Theorem (Theorem A) as a special case. To see that we take k=1 in corollary 2 and

$$b_{2n-1} = b_{3n-3} = ... = b_3 = b_1 = 0,$$

it follows that if $b_{2n} \ge b_{2n-2} \ge .... \ge b_2 \ge b_0 > 0$, then all the zeros of

$$F(z) = b_{2n}z^{2n} + b_{2n-2}z^{2n-2} + ... + b_2 z^2 + b_0$$

$$= b_{2n}(z^2)^n + b_{2n-2}(z^2)^{n-1} + ... + b_2(z^2) + b_0$$

lie in |z|≤1. Replacing z^2 by z and b_{2j} *by* b_j j=0,1,2....,n it followsthat if

$$a_n \ge a_{n-1} \ge .... \ge a_1 \ge a_0 > 0$$

then all the zeros of

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

lie in |z|≤1. which is precisely the conclusion of Eneström-Kakeya Theorem.

Taking k=2, in corollary 1 the following result follows ;

Corollary 3: if

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

is a polynomial of degree n≥2 such that either

$2a_n \ge a_{n-2} \ge .... \ge a_3 \ge a_1 > 0$ and $2a_n = a_{n-1} \ge a_{n-3} \ge .... \ge a_2 \ge a_0 > 0$,if n is odd

or

$2a_n \geq a_{n-2} \geq .... \geq a_2 \geq a_0 > 0$ and $2a_n = a_{n-1} \geq a_{n-3} \geq .... \geq a_3 \geq a_1 > 0$,

if n is even, then all the zeros of P(z) lie in

$$|z+1| \leq 2.$$

Next we prove the following generalization of Theorem C

Theorem 1.2: if

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_{2\lambda} z^{2\lambda} + ... + a_1 z + a_0,$$

is a polynomial of degree n≥2 such that either

$a_n \geq a_{n-2} \geq .... \geq a_{2\lambda+1} \leq a_{2\lambda-1} \leq ... \leq a_3 \leq a_1 > 0$
and $a_{n-1} \geq a_{n-3} \geq .... \geq a_{2\lambda} \leq a_{2\lambda-2} \leq ... \leq a_2 \leq a_0 > 0$,for some integer $\lambda,\ 0 \leq \lambda \leq \frac{n-1}{2}$, if n is odd, or

$a_n \geq a_{n-2} \geq .... \geq a_{2\lambda} \leq a_{2\lambda-2} \leq ... \leq a_2 \leq a_0 > 0$,and
$a_{n-1} \geq a_{n-3} \geq .... \geq a_{2\lambda+1} \leq a_{2\lambda-1} \leq ... \leq a_3 \leq a_1 > 0$, for some integer $\lambda,\ 0 \leq \lambda \leq \frac{n-2}{2}$ if n is even then all the zeroes of P(z) lie in the closed disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1} + 2(a_0 + a_1 - (a_{2\lambda} + a_{2\lambda+1}))}{a_n} \tag{7}$$

The following result is obtained by applying Theorem 1.2 to the polynomial P(tz):

Corollary 4: If

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

is a polynomial of degree n≥2 such that for some t>0 either

$a_n t^n \geq a_{n-2} t^{n-2} \geq .... \geq a_{2\lambda+1} t^{2\lambda+1} \leq a_{2\lambda-1} t^{2\lambda-1} \leq ... \leq a_3 t^3 \leq a_1 t > 0,$ and

$a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq .... \geq a_{2\lambda} t^{2\lambda} \leq a_{2\lambda-2} t^{2\lambda-2} \leq ... \leq a_2 t^2 \leq a_0 > 0,$ for some integer $\lambda,\ 0 \leq \lambda \leq \frac{n-1}{2}$,if n is odd

$a_n t^n \geq a_{n-2} t^{n-2} \geq .... \geq a_{2\lambda} t^{2\lambda} \leq a_{2\lambda-2} t^{2\lambda-2} \leq ... \leq a_2 t^2 \leq a_0 > 0,$ and

$a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq .... \geq a_{2\lambda+1} t^{2\lambda+1} \leq a_{2\lambda-1} t^{2\lambda-1} \leq ... \leq a_3 t^3 \leq a_1 t > 0,$ for some integer $\lambda,\ 0 \leq \lambda \leq \frac{n-2}{2}$,if n is even ,then all the zeros of P(z) lie in the closed disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq t + \frac{t^{n-1} a_{n-1} + 2(a_0 + a_1 t - t^{2\lambda}(a_{2\lambda} + a_{2\lambda+1} t))}{t^{n-1} a_n} \qquad (8)$$

2. Proofs of the Theorems

Proof of Theorem 1.1: consider

F(z) = (1-z²)P(z)

$$= -a_n z^{n+2} - a_{n-1} z^{n+1} + (a_n - a_{n-2}) z^n + ... + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z + a_0,$$

For |z|>1, we have

$$|F(z)| = \left| -a_n z^{n+2} - a_{n-1} z^{n+1} - k a_n z^n + a_n z^n + (k a_n - a_{n-2}) z^n + ... + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z + a_0 \right|$$

$$\geq |z|^n \left\{ \left| a_n z^2 + a_{n-1} z \ \ + (k-1) a_n \right| - \left| \begin{matrix} (k a_n - a_{n-2}) + (a_{n-1} - a_{n-3}) \frac{1}{z} + ... + (a_3 - a_1) \frac{1}{z^{n-3}} \\ + (a_2 - a_0) \frac{1}{z^{n-2}} + a_1 \frac{1}{z^{n-1}} + a_0 \frac{1}{z^n} \end{matrix} \right| \right\}$$

$$\geq \left| z^2 + \frac{a_{n-1}}{a_n} z + (k-1) \right| - \frac{1}{|a_n|} \left\{ \begin{array}{r} (ka_n - a_{n-2}) + (a_{n-1} - a_{n-3}) \frac{1}{|z|} + \ldots + (a_3 - a_1) \frac{1}{|z|^{n-3}} \\ + (a_2 - a_0) \frac{1}{|z|^{n-2}} + a_1 \frac{1}{|z|^{n-1}} + a_0 \frac{1}{|z|^n} \end{array} \right\}$$

$$> \left| z^2 + \frac{a_{n-1}}{a_n} z + (k-1) \right| - (k + \frac{a_{n-1}}{a_n})$$

>0,if

$$\left| z^2 + \frac{a_{n-1}}{a_n} z + (k-1) \right| > (k + \frac{a_{n-1}}{a_n})$$

Hence all the zeros of F(z) whose modulus is greater than 1 lie in the region

$$\left| z^2 + \frac{a_{n-1}}{a_n} z + (k-1) \right| \leq (k + \frac{a_{n-1}}{a_n}) \qquad (9)$$

But those zeros of F(z) whose modulus is less than or equal to 1 already satisfy the inequality(9).Since all the zeros of P(z) are also the zeros of F(z), therefore it follows that all the zeros of P(z) lie in the region(9).

Let α and β be the roots of the quadratic $z^2 + \frac{a_{n-1}}{a_n} z + (k-1) = 0$, therefore from(9) , we have $|z-\alpha||z-\beta| \leq k + + \frac{a_{n-1}}{a_n}$. which completes the proof of Theorem 1.1

Proof of Theorem 1.2: Consider

F(z) = $(1-z^2)$P(z)

$$= -a_n z^{n+2} - a_{n-1} z^{n+1} + (a_n - a_{n-2}) z^n + \ldots + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z + a_0$$

therefore for |z|>1, we have

$$|F(z)| = |-(a_n z^n + a_{n-1})z^{n+1} + (a_n - a_{n-2})z^n + ... + (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1 z + a_0|$$

$$\geq |z^{n+1}|\left\{|a_n z + a_{n-1}| - (|a_n - a_{n-2}|\frac{1}{|z|} + ... + |a_3 - a_1|\frac{1}{|z^{n-2}|} + |a_2 - a_0|\frac{1}{|z^{n-1}|} + |a_1|\frac{1}{|z^n|} + \frac{|a_0|}{|z^{n+1}|})\right\}$$

$$> |z^{n+1}|\{|a_n z + a_{n-1}| - (|a_n - a_{n-2}| + ... + |a_3 - a_1| + |a_2 - a_0| + |a_1| + |a_0|)\}$$

$$= |z^{n+1}|\left\{|a_n z + a_{n-1}| - (\sum_{j=0}^{n}|a_j - a_{j-2}| + |a_1| + |a_0|)\right\}$$

$$= |z^{n+1}|\left\{|a_n z + a_{n-1}| - (a_0 + a_1 + \sum_{k=1}^{\frac{n}{2}}|a_{2k} - a_{2k-2}| + \sum_{k=1}^{\frac{n-k}{2}}|a_{2k+1} - a_{2k-1}|)\right\} \quad (10)$$

Assuming first that n is even then from (10), for |z|=1, we have

$$|F(z)| > |z^{n+1}|\left\{|a_n z + a_{n-1}| - (a_0 + a_1 + \sum_{k=1}^{\lambda}|a_{2k-2} - a_{2k}| + \sum_{k=\lambda+1}^{\frac{n}{2}}|a_{2k} - a_{2k-2}| + \sum_{k=\lambda+1}^{\frac{n-2}{2}}|a_{2k+1} - a_{2k-1}|)\right\}$$

$$= |z^{n+1}|\{|a_n z + a_{n-1}| - 2(a_0 + a_1 - a_{2\lambda} - a_{2\lambda+1}) + a_n + a_{n-1}\}$$

>0,if

$$\left|z + \frac{a_{n-1}}{a_n}\right| > 1 + \frac{a_{n-1} + 2(a_0 + a_1 - a_{2\lambda} - a_{2\lambda+1})}{a_n} \quad (11)$$

In case n is odd it can be easily seen that |P(z)|>0 if (11) holds. Hence all those zeros of P(z) whose modulus is greater than 1 lie in the circle

$$\left|z + \frac{a_{n-1}}{a_n}\right| > 1 + \frac{a_{n-1} + 2(a_0 + a_1 - a_{2\lambda} - a_{2\lambda+1})}{a_n} \quad (12)$$

But all those zeros of P(z) whose modulus is less than or equal to 1 already satisfy (12). Therefore it follows that all the zeros of P(z) lie in the circle(12). which proves Theorem(1.2).

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